Imperial College London

MSc Thesis

Introduction to the AdS/CFT Correspondence

Author:
Maria Dimou

Supervisor:
Prof. Daniel Waldram

September 24, 2010

Contents

Introduction to the AdS/CFT Correspondence

Maria Dimou

September 2010

Abstract

This is a thesis of the Quantum Fields and Fundamental Forces MSc at Imperial College London, introducing the ADS/CFT correspondence, focused on the duality between $\mathcal{N} = 4$, $d = 4$ Superconformal Yang Miils theory and string theory on an $\text{AdS}_5 \times \text{S}^5$ background.

1 Introduction

The human persuit of gaining a deep understanding in the fundamental laws of nature, is what motivates the science of physics. The key starting point would, obviously, be the quest of a consistent way to describe the interactions between the elementary particles. The three basic theories that provide this description, are Quantum Electrodynamics, weak interactions and Quantum Chromodynamics. For the theory associated with the gravitational forces, however, there has not yet been found a consistent way of quantization. Gravity is non-renormalizable, thus, making the use of perturbative methods unsuccessful.

A rapidly developing physical theory that attempts to overcome that problem, is string theory, which first emerged in the early 1960's. Its fundamental objects are open and closed strings, which, when quantised, give rise to some massless modes and to a tower of massive particles. The fact that the spectrum of the closed string contains a massless mode with spin two, demonstrates that every string theory contains gravity, providing a way of its quantization. Quantum gravity was not the original motivation for string theory, though.

QCD is also problematic under perturbation theory because of its running coupling constant, that grows with the energy scale. At high energies, a perturbative expansion is well defined; at low energies, on the contrary, the strong coupling leads the associated elementary particles to form bound states with zero gauge color: the hadrons. In that sense, it is said that QCD is asymptotically free. A preceding way of describing hadrons at low energies, was through string theory. The idea had its origin in the string like behavior of the quark-antiquark flux tubes that are formed in the low energy dynamics of hadrons.

Such a connection between string theory and Yang Mills gauge theories, was proposed by 't Hooft [1], while seeking for a small parameter, in which perturbative expansions would be valid in the strong coupling regime. The suggested dimensionless parameter was the number of colors N, of $SU(N)$ gauge theories. For the conjecture to be applied in QCD, the expansion should be made in a way, such that the energy scale Λ_{QCD} would remain constant in the perturbative expansion of the β function. This lead to the choise: $\lambda = g_{YM}^2 N$, the 't Hooft coupling, is kept fixed, $N \to \infty$ and the expansion is performed in $1/N$. The graphical representation is in terms of Feynman diagrams in the double line notation and it is equivalent to the genus expansion of string world sheets. There is indeed such a mapping between Yang Mills gauge theories and string theory in some background, formulated in the AdS/CFT correspondence.

The original conjecture of the AdS/CFT correspondence was made by Maldacena [2] and stated that $\mathcal{N} = 4$ Superconformal Yang Mills theory in 4 spacetime dimensions is dual to Type II$_B$ string theory on $AdS_5 \times S^5$. In the diagrams mentioned above, each hole in a Riemann surface comes with a factor of g_s, in the string loop expansion, while, on the gauge theory side, it corresponds to a closed loop with two vertices and it comes with a factor of g_{YM}^2. This equivalence suggests that $g_{YM}^2 \sim g_s$ and that the expansion in $1/N$ corresponds to the string genus expansion. But what is the bouckground in which this string theory lives?

The dual gauge theory lives in 4 flat dimensions. String theories, on the other hand, only exist in 10 flat dimensions. We could perform a dimensional reduction by considerind string theory on $\mathbb{R}^4 \times M_6$, where M_6 is a compact manifold but strings are inconsistent in 4 flat spacetime dimensions. In fact, an attempt of string quantization requires an extra "Liouville" field and since the space where a agravitational theory lives has the same number of dimensions as the number of fields on the string, this extra field corresponds to an extra dimension, leaving us then, with 5 dimensions. A key point of the AdS/CFT correspondence, is the fact that the geometry of the boundary of the compactified AdS_5 space is 4-

dimensional Minkowski spacetime. The next step after this observation, would be the compactification of the 10-dimensional Type II_B string theory, in particular, of its low energy supergravity limt, on $AdS_5 \times S^5$. A way to think of Maldacena's conjecture is the following: A set of N coincidentl D3-branes is implemented into the 10-dimensional Type II_B supergravity. D3-branes are $(3+1)$-dimensional objects where open strings can end and they are charged under $(3+1)$-form gauge potentials. Their $(3+2)$-form field strengths belong to the supergravity multiplet and their flux contributes to the stress-energy tensor, so that the geometry becomes curved. Specifically, it turns out that the near horizon geometry is AdS_5, while the geometry far away from the branes remains flat. On the other hand, the D3-branes act as sources to the 10-dimensional Type II_B supergravity, they live on the 4-dimensional Minkowskian boundary of the compactified AdS_5 and their world volume is governed by a $\mathcal{N} = 4$ $SU(N)$ SYM theory, which arises from possible ways that open strings can stretch between the N branes.

For the correspondence to hold, the global symmetries of the two theories need to match. This is indeed the case, as the isometries $SO(2,4) \times SO(6)$ of $AdS_5 \times S^5$ are mapped into the $SU(2,2) \sim SO(2,4)$ group associated with the 4-dimensional Poincare symmetry, combined with the conformal symmetry times the $SU(4)_R \sim SO(6)_R$ R-symmetry group. Finally, the maximally supersymetric background $AdS_5 \times S^5$ has 32 supersymmetries, while the 16 supersymmetries of $\mathcal{N} = 4$ $SU(N)$ gauge theory are enhanced to 32 by adding the conformal supersymmetries.
Except of the global symmetries the correlation functions should also be mapped. This is realized by the establishing an 1-1 correspondance between the 5-dimensional fields living on AdS and the composite operators that specify the field theory spectrum. Then, we can think of the boundary values of the fields, as sources for the associated operators. In this content we identify the field theory's generating

functional with the on shell Type II_B effective action. This is expressed as:

$$e^{W_{CFT}[\phi_0]} = \langle e^{-\int d^4x \ \phi_0(x) O(x)} \rangle_{CFT} \simeq e^{S_{AdS_5}[\phi]}|_{\phi(boundary)=\phi_0}$$

and the correlation functions of the operators are given by:

$$\langle O_1 \cdots O_n \rangle = (-1)^{n-1} \frac{\delta}{\delta \phi_1^0} \cdots \frac{\delta}{\delta \phi_n^0} W_{CFT} \mid_{\phi_i^0 = 0}$$

Since, $\mathcal{N} = 4$ SYM is a conformal theory, there is no need for the 't Hooft parameter λ to be constant and we can consider further useful limit. We shall consider one very interesting limit in particular. When the radious R of the AdS_5 and the S^5 spaces is much larger than the string length, the curvature is small string theory is approximated by supergravity (low energy limit). Because of the relation $(R/l_s)^4 \sim \lambda$, the limit $\lambda \to \infty$ where supergravity is a good approximation corresponds to the regime where the field theory is strongly coupled, as the berturbative expansion in λ makes sense for $\lambda \ll 1$. This weak/strong duality is one of the most important consequence of the AdS/CFT correspondence.

2 Conformal Theories

The conformal group $C(1, n-1)$ in an n-dimensional Minkowski spacetime with a metric η, is the group of linear coordinate transformations that leave the metric invariant, up to a scale:

$$g'_{ab}(x) = \partial_a x'^c \partial_b x'^d \eta_{cd} = \Omega(x)\eta_{ab} \tag{1}$$

The conformal group is a symmetry of massless particles, preserving the structure of the light cone. For massive particles to be included, we need to impose the condition $\Omega = 1$, which leaves us with just the Poincare subgroup. $C(1, n-1)$ for $n > 2$ is finite and its killing vector is given by:

$$f^a = \alpha^a + \omega^{ba} x_b - \rho x^a + c_b(2x^a x^b - \eta^{ab} x^2) \tag{2}$$

where the first term corresponds to n translations, the second to $\frac{n(n-1)}{2}$ Lorentz transformations, the third to 1 dilatation and the last term in the brackets corresponds to n special conformal transformations.
By writing the killing vector as

$$f^a = \lambda_A \delta^A x^a \ , \qquad \lambda_A \equiv (a^a, \frac{1}{2}\omega^{ab}, \rho, c^a), \tag{3}$$

we find the corresponding infinitesimal coordinate transformations:

$$\delta^a_{\mathcal{T}} x^c = \eta^{ac} \qquad Translations \tag{4}$$

$$\delta^{ab}{}_{\mathcal{L}} x^c = \eta^{bc} x^a - \eta^{ac} x^b \qquad Lorentz \tag{5}$$

$$\delta_{\mathcal{D}} x^c = -x^c \qquad Dilatations \tag{6}$$

$$\delta^a_{\mathcal{C}} x^c = 2x^a x^c - \eta^{ac} x^2 \qquad Special\ Conformal \tag{7}$$

Their commutation relation define the Lie algebra of the conformal group. Introducing a basis:

$$G^A \equiv (p^a, j^{ab}, \Delta, k^a) \tag{8}$$

by the rule:

$$f = \lambda_A(\delta^A x_a)\partial_a = i\lambda G^A \tag{9}$$

we find the expressions:

$$\begin{aligned} ip^a &= \partial^a && (10) \\ ij^{ab} &= x^a\partial^b - x^b\partial^a && (11) \\ i\Delta &= x^a\partial_a && (12) \\ ik^a &= 2x^a x^c\partial_c - x^2\partial^a && (13) \end{aligned}$$

Instead of working in terms of the above differential realization, we can consider the faithful matrix representation and introduce the conformal algebra, requiring the basis:

$$G^A = (P^a, Jab = -J^{ba}, D, K^a) \tag{14}$$

that satisfies the same commutations relations as the basis 10-13. A generic element of the group is the form: $X = i\lambda_A G^A$ and the conformal group is obtained by considering the exponantiation: $g(\lambda) = e^{i\lambda_A G^A}$.

The conformal group has the following subgroups:

Scale+Translation:

$$[iD, D] = 0 \quad (15)$$

$$\left[iP^a, P^b\right] = 0 \quad (16)$$

$$[iD, P^a] = P^a \quad (17)$$

$$(18)$$

Poincare:

$$\left[iP^a, P^b\right] = 0 \quad (19)$$

$$[iJab, P^c] = \eta^{ac}P^b - \eta^{bc}P^a \quad (20)$$

$$\left[iJ^{ab}, J^{cd}\right] = \eta^{bc}J^{ad} - \eta^{ac}J^{bd} + \eta^{ad}J^{bc} - \eta^{bd}J^{ac} \quad (21)$$

Scale+Lorentz:

$$\left[iJ^{ab}, K^c\right] = \eta^{ac}K^b - \eta^{bc}K^a \quad (22)$$

$$[iD, K^a] = -K^a \quad (23)$$

$$\left[iK^a, K^b\right] = 0 \quad (24)$$

$$\left[iK^a, P^b\right] = 2\eta^{ab}D - 2J^{ab} \quad (25)$$

The algebra of $C(1, n-1)$ is isomorphic to the algebra of the Lorentz group $SO(2, n)$, which is a set of pseudoorthogonal transformations in a $(n{+}2)$-dimentional flat spacetime: M^{n+2} with metric: $\eta^{\mu\nu} = (-1, 1, \eta^{ab})$. Then, $(\mu, \nu) = -2, -1, 0, 1, ..., n-1$ and $(a, b) = 0, 1, ..., n-1$. Then, the $SO(2, n)$ algebra is given by:

$$[iM^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho}M^{\mu\sigma} - ...cycl.\ perms \quad (26)$$

If we define the generators:

$$D = M{-}2,{-}1\ ,\ J^{ab} = M^{ab}, \tag{27}$$
$$\frac{1}{2}(P^a - K^a) = M^{-2,a}$$
$$\frac{1}{2}(P^a + K^a) = M^{-1,a}$$

The, 26 gives the conformal algebra.

Two useful points to be stressed are: 1. In the interesting representation of the conformal group, fields that are eigenfunctions of the scaling operator D, with eigenvalues $-i\Delta$ and 2. Conformal symmetry sets certain constraints on the correlation functions of a CFT. In particular, the dimension of the 1-,2- and 3-point functions are fixed.

3 AdS Space

The n-dimensional Anti de Sitter space is the Lorentzian analogue of the n-dimensional hyperbolic space and constitutes the maximally symmetric vacuum solution of Einstein's equation in n dimensions, with a negative cosmological constant, Λ.

In vacuum, ($T_{\mu\nu} = 0$), the Einstein's equation becomes:

$$G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Lambda = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} g_{\mu\nu} \Lambda = 0 \tag{28}$$

By contracting with the metric, we find:

$$R = \frac{n}{n-2} \Lambda \tag{29}$$

and so,

$$R_{\mu\nu} = \frac{1}{n-2} g_{\mu\nu} \Lambda \tag{30}$$

Since the Ricci tensor is proportional to the metric, AdS$_n$ space is an Einstein space. By further requiring that:

$$R_{\mu\nu\rho\sigma} = \frac{2}{(n-1)(n-2)} \Lambda (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}), \tag{31}$$

the space is maximally symmetric.

For $n = p + 2$, AdS$_{p+2}$ can be identified as a $(p + 2)$-dimensional hyperboloid, which satisfies the equation:

$$x_0^2 + x_{p+2}^2 - \sum_{i=1}^{p+1} x_i^2 = R^2 \tag{32}$$

in a flat $(p+3)$-dimensional space $\mathbb{R}^{2,4}$, with metric:

$$ds^2 = -dx_0^2 - dx_{p+2}^2 + \sum_{i=1}^{p+1} x_i^2. \tag{33}$$

The isometry of this homogenous and isotropic space is by construction $O(2, p+1)$, the same as the one of the conformal group in $(p+1)$ dimensions. In the Euclidean version, the hyperboloid is represented as $x_0^2 - \sum_{i=1}^{p+2} x_i^2 = R^2$ in $\mathbb{R}^{1,5}$, with isometry $SO(1,5)$.

3.1 Global coordinates

One parametrization that solves equation 32 is the following:

$$\begin{aligned} x_0 &= R\ cosh\rho\ cos\tau \\ x_{p+2} &= R\ cosh\rho\ sin\tau \\ x_i &= R\ sinh\rho\ \Omega_i, \quad \sum_{i=1}^{p+1} \Omega_i^2 = 1 \end{aligned} \tag{34}$$

Then the metric becomes:

$$ds^2 = R^2(-cosh^2\rho\ d\tau^2 + d\rho^2 + sinh^2\rho\ d\Omega_p^2), \tag{35}$$

with Ω_p^2 being the line element of the p-sphere.

If we take $\rho \in [0, +\infty)$ and $\tau \in [0, 2\pi)$, the Minkowskian hyperboloid is covered once. By noticing that near $\rho = 0, ds^2 \approx R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega_i^2)$, which represents the disk $S^1 \times \mathbb{R}^{p+1}$, we observe closed timelike curves in the τ direction, corresponding to S^1. In order to obtain a causal spacetime, we need to unwrap that circle, by taking $\tau \in (-\infty, +\infty)$, which leaves us with a simply connected space, the universal covering of the hyperboloid. When we refer to AdS, we will always consider this universal covering.

3.2 Conformal Compactification

Using the global coordinates of Anti de Sitter space, it is easy to demonstrate the important fact, that the boundary of the conformally compactified AdS_n is the conformally compactified $(n-1)$-dimensional Minkowski space.
We start by looking at the conformal structure of flat Minkowski space, $\mathbb{R}^{1,p}$.

Conformal Compactification of $\mathbb{R}^{1,p}$

The metric of the $(p+1)$-dimensional Minkowski space is:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{p-1}^2, \tag{36}$$

where $d\Omega_{p-1}^2$ is the line element of the unit $(p-1)$-sphere. After the coordinate transformation $u_\pm = t \pm r$, we get:

$$ds^2 = -du_+ du_- + \frac{1}{4}(u_+ u_-) d\Omega_{p-1}^2 \tag{37}$$

Furthermore, by setting $u_\pm = tan\ \tilde{u}_\pm$ we obtain:

$$\begin{aligned} ds^2 &= \frac{1}{cos^2\ \tilde{u}_+\ cos^2\ \tilde{u}_-}(-d\tilde{u}_+ d\tilde{u}_- + \frac{1}{4} sin^2(\tilde{u}_+ - \tilde{u}_-) d\Omega_{p-1}^2 \\ &= \frac{1}{4\ cos^2\ \tilde{u}_+\ cos^2\ \tilde{u}_-}(d\tau^2 + d\theta^2 + sin^2\theta\ d\Omega_{p-1}^2), \end{aligned} \tag{38}$$

where $\tilde{u}_\pm = \frac{1}{2}(\tau \pm \theta)$.

The (t,r) half plane is now conformally mapped into the compact triangle in the (τ,θ) plane, with $\theta \in [0,\pi]$ and $\tau \in [-\pi,\pi]$.

After a conformal rescaling, we end up with:

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\ d\Omega_{p-1}^2, \tag{39}$$

which can be analytically continued to the maximally extended space, with $\theta \in [0, \pi]$, $\tau \in (-\infty, +\infty)$. This space has the geometry of the Einstein Static universe, $\mathbb{R} \times S^p$, with the north and south poles corresponding to $\theta = 0, \pi$. We have therefore conformally embeded $\mathbb{R}^{1,p}$ into $\mathbb{R} \times S^p$.

We saw that the conformal group of $\mathbb{R}^{1,p}$ is $SO(2, p+1)$, with maximal compact subgroup $SO(2) \times SO(p+1)$ and the generator of $SO(2)$ is $J_{0(p+2)} = \frac{1}{2}(P_0 + K_0)$. Now notice that the killing vector ∂_τ that corresponds to the global time translation on $\mathbb{R} \times S^p$, has the form

$$\partial_\tau = \frac{1}{2}(\partial_{u_+} + \partial_{u_-}) + \frac{1}{2}(u_+^2 \partial_{u_+} + u_-^2 \partial_{u_-}) = H \tag{40}$$

and can be identified with $J_{0(p+2)}$, with $P_0 = \frac{1}{2}(\partial_{u_+} + \partial_{u_-})$ and $K_0 = \frac{1}{2}(u_+^2 \partial_{u_+} + u_-^2 \partial_{u_-})$. Moreover, $SO(p+1)$ rotates the p-sphere. Consequently, the universal cover of the conformal group is identified with the isometry of the Einstein Static universe and the generator H ensures that the correlation functions of a conformal field theory on $(p+1)$-dimensional Minkowski space can be analytically extended to all of the Einstein Static universe.

Conformal Compactification of AdS$_{p+2}$

We now return to the universal cover of AdS$_{p+2}$, with the metric in the form 35, parametrized in the global coordinates (τ, ρ, Ω_i). We see a copy of the p-sphere plus a time $\tau \in (-\infty, +\infty)$. In this form, the $SO(p+1) \times SO(2)$ subgroup of the isometry $SO(2, p+1)$ of AdS$_{p+2}$ is manifest, with the $SO(2)$ goup corresponding to the translation in the time direction, with the killing vector $\partial\tau$ never vanishing and always well defined and the $SO(p+1)$ to the rotations of the sphere.

In order to show explicitly the causal structure of this space, let us use the new coordinate θ, for which $tan\ \theta = sinh\ \rho, \quad \theta \in [0, \pi/2]$.
The metric becomes:

$$ds^2 = \frac{R^2}{cos^2\theta}(-d\tau^2 + d\theta^2 + sin^2\theta\ d\Omega_p^2) \tag{41}$$

and after a conformal rescaling,

$$ds'^2 = (-d\tau^2 + d\theta^2 + sin^2\theta\ d\Omega_p^2). \tag{42}$$

This metric represents the geometry $\mathbb{R} \times S^{p+1}$. This time, however, $\theta \in [0, \pi/2]$, which means that the universal cover of the AdS$_{p+2}$ can be conformally mapped into one half of the Einstein Static universe. Each value of τ defines a $(p+1)$-dimensional hemisphere, with a boundary at the equator, which has the topology of a p-sphere. But, because $\tau \in (-\infty, +\infty)$, we have to introduce a boundary condition on the $\mathbb{R} \times S^p$ at $\theta = \pi/2$.
It is now obvious that the boundary $\mathbb{R} \times S^p$ at $\theta = \pi/2$ of the conformally compactified AdS$_{p+2}$ is the conformally compactified $\mathbb{R}^{1,p}$.

3.3 Poincare Coordinates

Another set of coordinates which parametrizes AdS, is given by a Lorentz vector plus a fifth coordinate $u > 0$, defined by:

$$\begin{aligned} x_0 &= \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 - t^2)) \\ x_{p+1} &= \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 + t^2)) \\ x_{p+2} &= R\ u\ t \\ x_i &= R\ u\ x_i. \end{aligned} \tag{43}$$

Then,

$$ds^2 = R^2(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2), \tag{44}$$

where we see the Minkowskian slices multiplied by u^2, which means that for an observer on a slice, all lengths appear to be rescaled by u. That is why these coordinates are called Poincare. The subgroups $SO(1,p)$ and $SO(1,1)$ are manifest here, where $SO(1,p)$ is the Poincare transformations on the slices, while $SO(1,1)$ is associated with the dilatation of the cobformal group and is realized as $(u, t, \vec{x} \to (\lambda u, t/\lambda, \vec{x}/\lambda)$.

The conformally equivalent metric $ds'^2 = \frac{ds^2}{u^2}$, has an $\mathbb{R}^{1,p}$ boundary at $u = \infty$, whereas $u = 0$ is a horizon, since the killing vector ∂_t becomes null. This is not a singularity, bacause these coordinates cover only half of the hyperboloid and we can extend the metric after the horizon with a different set of coordinates, like the global ones.

An additional useful form of the metric in Poincare coordinates is obtained by the redefinition:

$$u = \frac{1}{z} = e^r, \tag{45}$$

so that

$$ds^2 = R^2(\frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}) = R^2(dr^2 + e^2r(-dt^2 + d\vec{x}^2)) \tag{46}$$

and the boundary is at $z = 0$ or $r = \infty$ while the horizon is at $z = \infty$ or $r = -\infty$.

3.4 Euclidean Rotation

In field theory it is often convenient to perform a Wick rotaion to Euclidean signature. That way, the time-ordered correlation functions $\langle 0|T(\phi_1...\phi_n)|0\rangle$ of fields in Minkowski spacetime are related to the correlation functions $\langle \phi_1...\phi_n \rangle$ in the Euclidean space. The same can be done for fields in AdS space, for theories with a positive definite Hamiltonian [3, 4].

The Wick rotation expressed in the original coordinates is $x_{p+2} \to x_E = -ix_{p+2}$ and sends the equation 32 to

$$x_0^2 - x_E^2 - \vec{x}^2 = R^2 \tag{47}$$

and the equation 33 to

$$ds_E^2 = -dx_0^2 + dx_E^2 + d\vec{x}^2 \tag{48}$$

In the global coordinates, $\tau \to -i\tau$ and the metric 35 becomes

$$ds_E^2 = R^2(cosh^2\rho \; d\tau_E{}^2 + d\rho^2 + sinh^2\rho \; d\Omega_p^2), \tag{49}$$

wheres in the Poincare coordinates, we get instead of equation 44,

$$ds_E^2 = R^2(\frac{1}{u^2}du^2 + u^2(dt_E^2 + d\vec{x}^2), \tag{50}$$

by sending $t \to -it_E$.

Notice, that even though Poincare coordinates cover only half of the AdS space, the Euclidean rotation in t gives the same space as the Euclidean rotation in τ.

The Euclidean version of Minkowski $\mathbb{R}^{1,p}$ is $\mathbb{R}^{p+1}$, which can be compactified into the sphere S^{p+1}, by adding a point at infinity. So, the Euclidean version of the Minkowski slice at $u = \infty$, which constitutes the boundary of AdS_{p+2}, is a S^{p+1} sphere with one point removed; and the Euclidean AdS_{p+2} is mapped into a $p+2$-diamensional disk. The full $p+1$-spherical boundary is recovered by adding the $u = 0$ point that corresponds to spatial infinity; we see that the null plane $u = 0$ of the Lorentzian case, has now shrank into a point.

4 Superstring Theory and Supergravity

The correspondence of interest is between conformal $\mathcal{N} = 4 SU(N)$ Super Yang Mills field theory on one hand and Type II_B 10-dimensional string theory, compactified on $\mathrm{AdS}_5 \times \mathrm{S}_5$, on the other hand.

Let us begin considering the 10-dimensional flat case of Type II_B string theory, in which we have implemented N parallel D3 branes. This system has two types of perturbative excitations:

- closed string excitations, corresponding to excitations of the empty space; in the low energy limit, only the massless states are excited, giving a gravity supermultiplet in 10 dimensions and the associated effective Lagrangian is the Type II_B Supergravity one.
- open string excitations, corresponding to the excitaions of the D-branes; the low energy massless spectrum falls in a $\mathcal{N} = 4$ vector supermaltiplet in (3+1) dimensions and the associated Lagrangian is the one of $\mathcal{N} = 4U(N)$ Super Yang Mills field theory [5].

The coupled effective action of the above system, has the the form:

$$S = S_{bulk} + S_{brane} + S_{int.} \tag{51}$$

where S_{bulk} corresponds to the 10-dimensional Supergravity, S_{brane} is defined on the (3+1)-dimensional brane worldvolume and $S_{int.}$ describes the interactions between the excitations of the brane and those of the bulk. The action of the bulk can be expanded as free massless modes propagating, plus some interaction terms. The brane action can be expanded in higher derivative terms as well. We will see that, by taking the low energy limit $(\alpha', l_s \to 0)$, all the interaction terms in S_{bulk}, the higher derivative terms in S_{brane} and the whole of $S_{int.}$ vanish, leaving us with two decoupled systems:

- free supergravity in the 10-dimensional bulk
- $\mathcal{N} = 4U(N)$ conformal Super Yang Mills gauge theory on the (3+1) dimesional world wolume of the D3-branes.

These D3-branes are a special case of the Dp-branes, which are extended in $(p+1)$ spacetime dimensions massive objects, whose world volume is governed by a $U(N)$ gauge theory at low energy. They are charged under a $(p+1)$-form gauge potentials and they can be thought of as sourses for the II_B supergravity background. Their $(p+2)$-form gauge field strenghts belong to the supergravity multiplet and their flux, generated by implementing the branes in the background, contributes to the stress-energy tensor, causing the geometry to curve.
We can identify the Dp-branes with the extremal solution of supergravity and by looking at the weak coupling limit, where $g_s \to 0$, we see that the metric turns out to be everywhere flat, except on the $(p+1)$-dimensional hyperplane. The result is just a localised defect in the flat spacetime. For a D3-brane solution of supergravity in particular [6], the geometry near the horizon (near the brane) becomes $\text{AdS}_5 \times S^5$. Strings propagate free in the flat spacetime, untill they reach the branes. The decoupled systems now are:

- free supergravity in the 10-dimensional bulk
- low energy superstring theory on $\text{AdS}_5 \times S^5$.

Comparing the two forms of the decoupled systems and noticing that the first one is the same in both cases, we can make the conjecture that $\mathcal{N} = 4U(N)$ Super Yang Mills field theory in (3+1)-dimensions is dual to Type II_B superstring background $\text{AdS}_5 \times S_5$ and lies on the boundary of AdS_5, which, as we saw, is Minkowski spacetime.

Both theories have the same isometries: AdS_5 is by definition symmetric under $SO(4,2)$, that is isomorphic to the conformal group in (3+1) dimensions, under which the field theory is invariant. The $SO(6)$ symmtry of rotations of the five-sphere can be identified with the $SU(4) \approx SO(6)$ R-symmetry group that rotates the six scalars and the four fermions of the field theory. Finally, string theory on $\mathrm{AdS}_5 \times S_5$ has 32 supercharges, while, from the field theory side, the 16 supercharges of the Yang Mills, are enhanced to 32, by adding the conformal supersymmetries.

4.1 Maximal Supersymmetries

Strings with tension $T_s = \frac{1}{2\pi\alpha'}$, where α' is the Regge slope with units of spacetime length squared, when quantised, give rise to some massless modes and to a tower of particles that have masses $m^2 \sim \frac{1}{\alpha'}$. There is always the posibility for the two endpoints of an open string to join, giving a closed string. Therefore, closed strings are part of every string theory. The massless closed string spectrum contains a rank-2 tensor, which can be decomposed into a symmetric tensor $g_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and a scalar field ϕ. That massless, spin-2 mode is identified with the graviton and thus, every string theory contains gravity and provides a consistent way of its quantization.

For string theory to be defined and treated with perturbative methods in a flat space, the space needs to be 10-dimensional. 10-dimensional string theory is supersymmetric, and at low energies, the effective action of the closed string's massless modes reduces to supergravity. We are interested in representations of supersymmetry in which the graviton is massless and belongs to a massless supermultiplet of states and fields with spin ≤ 2. This inequality leads to restrictions on the maximal number of supercharges $\mathcal{N}$ that we can have in various dimensions.

Massless unitary supersymmetry representations require vanishing central charges and the anticommutation relations of the fermionic supersymmetry generators reduce to [7, 8]:

$$\{Q^a_\alpha, (Q^b_\beta)^\dagger\} = 2(\Gamma_\mu)_\alpha^\beta P^\mu \delta_a^b \ , \qquad \{Q^a_\alpha, Q^b_\beta\} = 0 \tag{52}$$

where Γ_μ are the standard Clifford-Dirac matrices and $a, b = 1, ..., \mathcal{N}$. For the massless representaions, we go to the frame $P_\mu = (E, 0, ..., 0, E), E > 0$ and the above equation becomes:

$$\{Q^a_\alpha, (Q^b_\beta)^\dagger\} = 2\delta_a^b \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_\alpha^\beta \tag{53}$$

We see that half of the supercharges vanish: $Q^a_\alpha = 0 \ , \ \alpha = \frac{1}{2}dimS + 1, ..., dimS$, where S is the Dirac spinor representation, with complex dimension $dim_{\mathbf{C}}S = 2^{[\frac{d}{2}]}$. The rest split into raising and lowering operators that alternate the helicity by 1/2. The total number of raising operators is $\frac{1}{2}\frac{1}{2}\mathcal{N}dim_{\mathbf{R}}S$. Since total helicity takes values from -2 to 2 with steps of 1/2, the maximum number of raising operators is 8. Consequently,

$$\mathcal{N}dim_{\mathbf{R}}S \leq 32, \tag{54}$$

which means that the maximum possible number of supercharges is 32.

E. Cremmer, B. Julia and J. Scherk [9] discovered a supergravity theory with $\mathcal{N} = 1$ in d=11 dimensions,the largest number of dimensions that saturates the bound 54, which is unique and has 32 Majorana supercharges. Theories of lower dimensions can be constructed via Kaluza Klein compactification of the d=11 theory. The string theories with maximal supersymmetry that contrain closed as well as open strings, with their associated Dp-branes, but no tachyon, have $\mathcal{N} = 2$ in 10 dimensions. Namely, Type II$_A$ (for p even) and Type II$_B$ (for p odd) string theories. We shall be interested in the second one and in its lower energy supergravity limit in particular.

4.2 Type II$_B$ Supergravity

The $\mathcal{N} = 2$, d=10 Type II$_B$ supermultiplet consists of the following fields:

Table 1: II$_B$ Supergravity field content

Notation	dof	field
$G_{\mu\nu}$	35_B	Metric - Graviton
$C + i\Phi$	2_B	Axion - Dilaton
$B_{\mu\nu} + iA_{2\mu\nu}$	56_B	rank-2 antisymmetric
$A_{4\mu\nu\rho\sigma}$	35_B	rank-4 antisymmetric
$\psi^a_{\mu\alpha}$, $a = 1, 2$	112_F	Majorana-Weyl Gravitinos
λ^a_α , $a = 1, 2$	16_F	Majorana-Weyl Dilatinos

The two gravitinos have opposite chirality with respect to the two gravitinos and, in that sense, the theory is parity violating (chiral). The fact that the field strength $F_5 = dA_4$ of the rank-4 antisymmetric tensor is required by sypersymmetry to be self-dual, gives rise to difficulties in writing a covariant action for this theory that would yield all equations of motion. What we can do, is to write an action with both dualities of A_4, supplemented with the self-duality condition

$$\tilde{F}_5 = *\tilde{F}_5. \tag{55}$$

There have been attempts to solve that problem, by writing "locally defined covariant" actions [10, 11]. However, the action usually used is [14, 12, 13]:

$$S_{II_B} = \frac{1}{4\kappa_B^2}\int dx^{10}\sqrt{G}e^{-2\phi}\left(2\mathcal{R} + 8(\partial\phi)^2 - |H_3|^2\right) -$$
$$-\frac{1}{4\kappa_B^2}\int dx^{10}\sqrt{G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) +$$
$$+\frac{1}{4\kappa_B^2}\int dx^{10}\left(A_4 \wedge H_3 \wedge F_3\right) \ + \ fermions \tag{56}$$

where $\sqrt{G} = \sqrt{-det\ G_{\mu\nu}}$, $\int \sqrt{G}|F_p|^2 \equiv \frac{1}{p!}\int \sqrt{G}G^{\mu_1\nu_1}\cdots G^{\mu_p\nu_p}\bar{F}_{\mu_1\cdots\mu_p}F_{\nu_1\cdots\nu_p}$ and the field strengths are defined as:

$$\begin{aligned} F_1 &= dC \quad , \quad H_3 = dB \quad , \quad F_3 = dA_2 \quad , \quad F_5 = dA_4 \\ \tilde{F}_3 &= F_3 - CH_3 \quad , \quad \tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3 \end{aligned} \tag{57}$$

This is the low energy effective action of string theory restrected to massless modes. It contains the parameter α', $(2{\kappa_B}^2 = (2\pi)^7(\alpha')^4)$, which is related to the string length as $l_s = \sqrt{\alpha'}$ and the parameter g_s, which is determined by the vacuum expectation value of the dilaton field.

$$g_s = \langle e^{\phi}\rangle \tag{58}$$

The first one determines the string tension $T_s = \frac{1}{2\pi\alpha'}$ and the masses of the string modes, $m^2 \sim \frac{1}{\alpha'}$, while the second determines the string coupling constant, controlling string interactions and quantum corrections. They can be used to correct the above action, as the effect of integrating out the massive modes gives higher derivative corrections, proportional to powers of α', while quantum corrections involve arbitary number of derivatives, weighted by powers of g_s. Thus, a physicsl quantity can be expanded as:

$$\sum_{g=0}^{\infty} g_s^{2g-2} f_g\left(\frac{\alpha'}{R^2}\right) \tag{59}$$

Perturbative expansion, which makes sense in the weak coupling limit $g_s \to 0$, is represented with Feynman graphs that constitute of Riemann surfaces with a number of holes (loops), equal to the genus g. Each order corresponds to one only graph, which is calculable and results in a non-trivial function of α'.

Type II$_B$ supergravity is invariant under the group $SU(1,1) \sim SL(2, \mathbf{R})$ and in order to make that manifest, we introduce the complex objects:

$$\tau \equiv C + ie^{-\phi} \quad , \quad G_3 \equiv \frac{1}{\sqrt{Im\tau}}(F_3 - \tau H_3), \tag{60}$$

by using, instead of the string metric $g_{\mu\nu}$, the Einsten metric $g_{E\mu\nu} \equiv e^{-\phi/2} g_{\mu\nu}$. Then,

$$S_{II_B} = \frac{1}{4\kappa_B^2} \int dx^{10} \sqrt{G_E} \left(2\mathcal{R}_E - \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{(Im\tau)^2} - \frac{1}{2}|F_1|^2 - |G_3|^2 - \frac{1}{2}|\tilde{F}_5|^2 \right) - \\ - \frac{1}{4i\kappa_B^2} \int dx^{10} \left(A_4 \wedge \bar{G}_3 \wedge G_3 \right) \; + \; fermions \tag{61}$$

The dilaton-axion field is subjected to a Mӧbius transformation

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \; , \quad ad - bc = 1 \; , \; a, b, c, d \in \mathbf{R} \; , \tag{62}$$

$B_{\mu\nu}, A_{\mu\nu}$ rotate into each other as $G_3 \to G_3' = \frac{c\bar{\tau}+d}{|c\tau+d|} G_3$
while the metric and the 4-form remain invariant.

4.3 p-brane Solutions in Supergravity

The equations of motion of the Type II$_B$ supergravity action, are satisfied by a particular set of objects that are invariant under the group $\mathbb{R}^{p+1} \times SO(1,p) \times SO(9-p)$ and preserve half of the supersymmetries. They are massive objects, extended in p-spacelike directions and charged under a $p+1$-gauge potential. They couple to the supergravity action by:

$$S_{p-brane} = T_p \int dx^{p+1} \sqrt{g} \quad + \quad q_p \int dx^{p+1} A_{p+1} \tag{63}$$

These solutions are called black p-branes [6, 15], as they are a generalization of rotating black holes. The first part of the above action defines their tension T_p and the second, their charge q_p. A more useful quantity than the charge itself, is its flux on a $(8-p)$-sphere surrounding the source.

$$\int_{S^{8-p}} *F_{p+2} = N \tag{64}$$

By the Dirac quantization condition, the flux N takes integer values and its relation with the charge is given by $N = q_p (2\pi)^p g_s (\alpha')^{(p+1)/2}$. The tension of the brane is identified with the energy per unit volume, in the p-spacelike directions. As a rotating black hole solution, it contains naked singularities, which are avoided by imposing the condition for the energy density to be greater than the charge.

$$E = T_p \geq \frac{N}{(2\pi)^p g_s (\alpha')^{(p+1)/2}} \tag{65}$$

This is the BPS bound for the 10-dimensional supersymmetry and the solutions that saturate it are called extremal p-branes. They preserve half of the 32 supersymmetries of Type II_B supergravity, i.e. they are 1/2 BPS objects. The associated solution is:

$$ds^2 = H^{-1/2}(r) dx_\mu dx^\mu + H^{1/2}(r) dy^2 \tag{66}$$

$$A_{0\ldots p} = H(r) \tag{67}$$

$$e^\phi = g_s H(r)^{(3-p)/4} \tag{68}$$

$$H(r) = 1 + \frac{C g_s N (\alpha')^{(7-p)/2}}{r^{7-p}} \tag{69}$$

where $x_\mu, \mu = 0, ..., p$ are the coordinates of the $(p+1)$-dimensional world volume of the brane, y_i, with $i = 1, ..., 9-p$, are the directions transverse to the brane and $r^2 = y_i y_i$ is the radial distance. The horizon collapses on the singularity, located at $r = 0$. The supergravity equations of motion are satisfied for 66 and 68 for any harmonic in the transverse space function $H(y_i)$. So, we can generalize 69, which represents a single extremal p-brane of charge N, located at the point $\vec{y} = 0$, by considering the function:

$$H(y_i) = 1 + C g_s (\alpha')^{(7-p)/2} \sum_{a=1}^{N} \frac{1}{|y - y_a|^{7-p}} \tag{70}$$

This is the multicentered solution, corresponding to N parallel extremal p-branes of unit charge, located at the ponts $\vec{y}_a$. The fact that the multicenter solution still saturates the BPS bound, means that the gravitational attraction of the extremal p-branes compensates their gauge repulsion, as the tension is equal to the charge by equation 65. Then, the energy of a system of extremal p-branes, equals the sum of the energies of the single branes and the potential energy is zero. Thus, the p-branes can get seperated or be brought closer to each other with no energy cost.

This classical supergravity description of black p-branes makes sense when the curvature of the brane geometry is much smaller than the string scale and also, the effective string coupling e^ϕ is also small, such that string loop corrections may be neglected.

Apart from the background fields of the II_B supergravity multiplet, p-branes have additional fields living on their world volume. Let us consider flactuations in the transverse directions to a p-brane. By writing the Nambu-Goto action in terms of the induced metric on the world volume of the brane, allowing variations of the transverse positions and then expanding in power series, keeping the terms up to two derivatives, we recognise a term corresponding to the standard kinetic terms for $9-p$ scalar fields. We can argue then, that there are $9-p$ scalar fields living on the world volume of the p-brane and they parametrize its position in spacetime. Figuring out what other fields could be living on the brane, is straight farward for the case of the extremal p-branes. As 1/2 BPS objects, preserving half of the supersymmetries of the background, they should have $\frac{1}{2}32 = 16$ supercharges. And the only supermultiplet with 16 supercharges, in any dimensions, is the vector supermultiplet, which indeed includes $9 - p$ scalar fields. Thus, supersymmetry fixes the fields on the world-volume of the extremal p-brane as well as their effective Lagrangian, up to two derivatives.

4.4 D-branes in String Theory

When p-branes are to be extended to solutions of the full Type II_B String theory, they will be subject to α' corrections. We see from equation 69, that in the weak coupling limit $g_s \to 0$, where perturbation theory can be applied in string theory, the metric is everywhere flat, except on the $(p+1)$-dimensional hypersurface of the brane, where it becomes singular. Strings propagate in the flat 10-dimensional spacetime and when they reach the deformation yielded by a brane, the interaction taking place is characterised by suitable boundary conditions on the string dynamics.

A string is parametrized by a spatial coordinate $\sigma \in [0, \pi]$. In ortder to extremize the Polyakov action and get the equations of motion, we need to set to zero the boundary contribution of the total derivative: $\partial_\sigma X^\mu \delta X_\mu$ at $\sigma = 0, \pi$. The appropriate conditions for this to be satisfied are:

- Neuman boundary conditions: $\partial_\sigma X^\mu = 0, \; \sigma = 0, \pi$, which means that the end points of an open string are allowed to move freely on the hypersurface of the brane, and

- Dirichlet boundary conditions: $\delta X^\mu = 0,\ \sigma = 0, \pi$, which means that the end points are not allowed to become detached from the brane.

Imposing the above conditions on two different set of coordinates

$$\begin{aligned} \partial_\sigma X^\mu = 0 \ , \quad \mu = 0, ..., p+1 \\ X^i = const \ , \quad i = p+1, ..., 10, \end{aligned} \tag{71}$$

we fix the end ponts of the open strings to move on a $(p+1)$-dimensional hypersurface of the 10-dimensional spacetime. The Poincare group $\mathbb{R}^{10} \times SO(1,9)$ brakes then to a lower dimensional one, $\mathbb{R}^{p+1} \times SO(1,p)$, defined on this hypersurface, times the $SO(10-p-1)$ group of rotations in the transverse space.
In string theory, these hypersurfaces that provide boundary conditions for the open strings' dynamics, are called Dp-branes (Dirichlet branes) [16, 17, 18]. They can be thought of as stringy solitons that carry Ramond-Ramond charges and provide an alternative description of the extremal p-branes in supergravity.

Quantization of open strings ending on D-branes, gives rise to the first, massless, excited states, which are classified as the either the longtudinal to the brane string oscillations, corresponding to a gauge field $A_\mu,\ \mu = 0, ..., p+1$ or the transverse ones, corresponding to scalar fields $\Phi_i,\ i = p+1, ..., 10$. The scalars are interpreted as fluctations of the brane in the transverse space with $SO(9-p)$ isometry, under which they transform as a vector. We also obtain their fermionic superpartners, which we shall ignore. Higher excited states, give a tower of massive string modes, with $m^2 \sim \frac{1}{\alpha'}$.

The vector field and the $9-p$ scalars form a $U(1)$, abelian, massless vector supermultiplet with 16 supercharges, as expected. At leading order, their effective action can be obtained by dimensional reduction of the 10-dimensional $U(1)$ Super Yang Mills gauge theory. Taking into account the higher order corrections, determined by the string perturbative expansion of open and closed strings, the world volume action plus interactions with the background is:

$$\begin{aligned} S_p = & - T_p \int d^{p+1}\sigma \left(e^{-\phi} \sqrt{-det\left(P[G+B]_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} \right) + \\ & + q_p \int d^{p+1}\sigma \ P\left[\sum A_{p+1} e^{B+2\pi\alpha' F} \right] \end{aligned} \tag{72}$$

Where the first part is the Born-Infeld action [19, 20] and describes the coupling of the world volume fields to the massless Neveu-Schwarz fields G_{ab}, Φ and B_{ab}, the second part is the Wess-Zumino action that describes the coupling to the massless background Ramond-Ramond forms and $T_p = q_p = 1/(2\pi)^p g_s (\alpha')^{(p+1)/2}$.

For defying the pull-backs $P[...]$ of the background tensors to the world volume of the brane, we first define the world volume by setting:

$x^i = 0$, for $i = p+2, ..., 10$

and then we map the coordinates parametrinzing the world volume, to the space-time coordinates corresponding to that hypersurface:

$\sigma^a = x^a$, for $a = 0, ..., p+1$.

The metric pull-back then becomes:

$$\begin{aligned} P[G_{ab}] &= g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b} \\ &= G_{ab} + (G_{ia}\partial_b + G_{ib}\partial_a)x^i + G_{ij}\partial_a x^i \partial_b x^j \\ &= G_{ab} + 4\pi\alpha'(G_{ia}\partial_b + G_{ib}\partial_a)\Phi^i + 4\pi^2\alpha'^2 G_{ij}\partial_a\Phi^i\partial_b\Phi^j \end{aligned} \tag{73}$$

We now see the already mentioned kinetic terms of $9-p$ scalar fields Φ^i, which are identified with

$$x^i = 2\pi\alpha'\Phi^i, \tag{74}$$

so that they get dimensions of $[\text{length}]^{-1}$ and describe the displacements of the brane in the transverse directions. The background fields in the action 79 are functionals of these scalar fields.

4.5 Gauge Field Theories on D-Branes

As we saw, equation 70 describes a set of N parallel D-branes with unit charge, located at the points $\vec{y}_a$, whereas equation 69 referes to a single D-brane of charge N. We can actually think of the latter, as the limit of the more generic function 70, in which all the branes coincide. The vector supermultiplets that live on the D-branes originate from the quantization of open strings that end on their hypersurfaces. When both ends of such a string are attached to the same brane, its length can be arbitarily small, resulting in a massless supermultiplet that corresponds to an abelian $U(1)$ gauge theory. The symmetry of a set of N identical parallel D-branes will be $U(1)^N$. However, there is also the possibility of the string stretching between two different branes. Then its length is bounded by the branes' distance and the corresponding multiplets will have massive modes, with $m \sim |\vec{y}_1 - \vec{y}_2| / \alpha'$. Now, when the branes come close enough to one another, so as to be thought of as coincident, the masses of these multiplets become zero and the gauge symmetry of the set is enhanced to $U(N)$, with N^2 vector fields, associated with N^2 possible massless strings. By ignoring thel factor $U(1)$ that corresponds to the overall positions of the coincident branes, we are left with a pure $SU(N)$ Yang Mills theory on $(p+1)$ dimensions [21].

Inversely, a massless vector field coming from a quantised string connecting two different coincident branes, will gain mass once we seperate the branes and the corresponding gauge symmetry $U(2)$ will be spontaneously broken into $U(1) \times U(1)$.

This mechanism described above from the point of view of D-branes in string theory or, equivalently, extremal p-branes as supergravity solutions, can also be described in terms of gauge field theories in $(p+1)$ dimensions.

As mentioned in the previous section, one can deduce the action of the maximally supersymmetric $(p+1)$-dimensional gauge theory through the dimensional reduction of the 10-dimensional theory. The bosonic part of the latter is:

$$-\int dx^{10} Tr[F_{\mu\nu}F^{\mu\nu}] \ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right] \tag{75}$$

When lowering the dimensions to (p+1), the 10-dimensional vector is divided into a vector A_a, $a = 0, ..., p+1$ and $9-p$ adjoint scalars Φ^i, $i = p+2, ..., 10$, all represented as $N \times N$ hermitian matrices. The boundary conditions force the derivatives with respect to the transverse directions to vanish. Then, the part of the action associated with the scalar fields becomes:

$$\int dx^{p+1} \sum_{i=p+2}^{10} Tr\left(\partial_a \Phi^i + i\left[A_a, \Phi^i\right]\right)^2 \ + \ \sum_{i,j=p+2}^{10} Tr\left[\Phi^i, \Phi^j\right]^2 \tag{76}$$

For the supersymmetry to be respected by the vacuum, the minimum of the scalar potential must be zero, i.e. $[\Phi^i, \Phi^j] = 0$. Therefore, the matrices representing the scalar fields can be simultaneously diagonalized and their vacuum expectation values take the form:

$$\Phi^i = diag\left(\Phi^i_1, ..., \Phi^i_N\right) \tag{77}$$

Then, the term $[A_a, \Phi^i]^2$ gives

$$\left|A^{kl}_a\right|^2 \sum_{i=p+2}^{10} \left|\Phi^i_k - \Phi^i_l\right|^2, \tag{78}$$

which shows that the diagonal components of the $N \times N$ matrix A_a are massless, while the off-diagonal ones aquire a mass, proportional to $|\Phi^i_k - \Phi^i_l|$, $k \neq l$.

The correspondance is obious by considering that the vacuum expectation values of the scalar fields defines the positions of the N D-branes in spacetime, as in equation 81. In the case where all the eigenvalues $\Phi^i_1, ..., \Phi^i_N$ are equal, all of the N gauge fields remain massless and they are obtained by the quantization of open strings, each one of them having both end points on the same D-brane; the associated symmetry is $U(1)^N$. If n of the eigenvalues are distinct, then the symmetry is $U(1)^{N-n} \times U(n)$, with n massive and $N - n$ massles gauge fileds, coming from the quantization of strings streching between these n branes that were seperated from the rest of the system.

4.6 D3-Branes

Let us now focus on the specific case of D3-branes. For $p = 3$, the solution 66-69 to the supergravity equations of motion becomes:

$$
\begin{aligned}
ds^2 &= H^{-1/2} dx_\mu dx^\mu + H^{1/2}(dr^2 + r^2 d\Omega_5^2) && (79)\\
A_4 &= H(r) && (80)\\
e^\phi &= g_s && (81)\\
H(r) &= 1 + \frac{C g_s N (\alpha')^2}{r^4} && (82)
\end{aligned}
$$

This particular solution has the following properties:

- The world volume of the brane has a 4-dimensional Poincare symmetry $\mathbb{R}^4 \times SO(1,3)$.
- The dilaton and the axion fields are constant.
- The geometry is regular at $r = 0$.
- It is self dual (the magnetic dual of the D3-brane is also a D(10-4-3)=D3-brane).

For a set of N parallel D3-branes, located at the points y_i, the solution is:

$$H(y_i) = 1 + C g_s (\alpha')^2 \sum_{a=1}^{N} \frac{1}{|y - y_a|^4} \qquad (83)$$

We define the radius R of the brane as:

$$R^2 = \alpha' \sqrt{C g_s N} \qquad (84)$$

When the radius is large comparing to the string scale α', i.e. $g_s N \gg 1$, the supergracity low energy limit constitues a reliable approximation to the full string theory. Even when we take N to be large, we can demand $g_s \ll 1$, so that perturbative methods can be applied.

Using the parameter R, the metric 79 can be rewritten as:

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} dx_\mu dx^\mu + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2) \tag{85}$$

We can study the geometry in to interesting limits:

- $r \gg R$: In this case, when studying the geometry far from the brane, we see that the deformation vanishes and we recover flat 10-dimensional spacetime.
- $r \ll R$: The metric near the brane becomes:

$$ds^2 \equiv \frac{r^2}{R^2} dx_\mu dx^\mu + R^2 \frac{dr^2}{r^2} + r^2 d\Omega_5^2 \tag{86}$$

  which corresponds to the product of two Einstein spaces: $AdS_5 \times S^5$.

5 The Maldacena AdS/CFT Correspondence

In the previous sections, it was shown that on the world volume of Dp-branes, implemented in a Type II$_B$ string theory background, live (p+1)-dimensional gauge theories. For the case p=3 in particular, the world volume theory is $\mathcal{N}=4,\ d=4$ Superconformal Yang Mills, while the theory near the brane horizon is Type II$_B$ string theory (or supergravity) on $AdS_5 \times S^5$. We are now about to describe the AdS/CFT (Anti-de-Sitter/Conformal Field Theory) correspondence, that examines the equivalences between these two theories.

5.1 The Maldacena Limit

Maldacena's conjecture about the $\mathcal{N}=4,\ d=4\ SYM \leftrightarrow II_B$ string theory on $AdS_5 \times S^5$ duality was inspired by the fact that each of the two theories decouples from the system in which it is considered, in the same limit: $\alpha' \to 0$.

On one hand, we consider the $\mathcal{N}=4,\ d=4\ SYM$ theory living on the world volume of N parallel D3-branes, implemented in a Type II$_B$ string background. Performing an expansion in α' of the effective action for the coupled system of the branes and the bulk, we get the leading terms:

$$\frac{1}{g_s}\int dx^4 F_{\mu\nu}^2 + \frac{1}{\alpha'^4}\int dx^{10}\sqrt{G}\mathcal{R}e^{-2\phi} + ... \tag{87}$$

where the coupling constant is fixed by the vacuum expectation value of the dilaton. It is obvious that when $\alpha' \to 0$, the theory on the brane decouples from the theory in the 10-dimensional bulk.

On the other hand, when considering the D3-brane solution of supergravity, in the limit $\alpha' \to 0$, the decoupled system is again the 10-dimensional bulk and the

near horizon geometry of $AdS_5 \times S^5$, the so called "throat". This limit however, should be approached in a cosnistent way, such that the physical observables are kept finite. More specifically, as mentioned in section 4.5, the mass of a gauge vector field is given by the mass of an open string, stretching between two branes. That is: $m \sim \frac{\Delta r}{\alpha'} \sim \delta\Phi$ (remember that the transverse positions of the brane are parametrized by the vacuum expectation values of the SYM scalar fields).
We want to keep this quantity fixed and so, we define a new variable $u \equiv r/\alpha'$ The correct way to take the Maldacena limit then, is:

$$\begin{aligned} \alpha' &\to 0 \\ r &\to 0 \\ g_s &: fixed \\ N &: fixed \\ u \equiv \frac{r}{\alpha'} &: fixed \end{aligned} \tag{88}$$

The new form of the metric 86, in the near horizon limit, is:

$$ds^2 \simeq \alpha' \left[\frac{u^2}{\sqrt{Cg_sN}}\, dx_\mu dx^\mu + \sqrt{Cg_sN}\, \frac{du^2}{u^2} + \sqrt{Cg_sN}\, d\Omega_5^2 \right] \tag{89}$$

The fact that the spacetime $AdS_5 \times S_5$ deformation caused by the branes decouples from the 10-dimensional flat bulk, as $\alpha' \to 0$, can be seen from the function 83, which we now write as:

$$H(r) = 1 + \frac{Cg_sN}{\alpha'^2 u^4} \tag{90}$$

This can alternatively be thought of as zooming in on the $N = 4,\ d = 4$ conformal Super Yang Mills field theory on the D3-world volume, by taking $r \to 0$.

It therefore, makes sense to propose the duality between $N = 4$ SYM and Type II_B superstring theory or supergravity on $AdS_5 \times S_5$ background.

Let as now state the relations between the parameters of the two theories.

▲ Comparing the $N = 4$ SYM action

$$S = \frac{1}{g_{YM}^2} \int d^4x Tr \left(-\frac{1}{2} F_{\mu\nu}^2 - i\bar{\psi}^a \gamma_\mu D^\mu \psi_a - (D_\mu \Phi^i)^2 + c_i^{ab} \psi_a [\Phi^i, \psi_b] + [\Phi^i, \Phi^j]^2 \right)$$

and the kinetic term of the brane/bulk coupled effective action for the gauge fields $S \sim \frac{1}{g_s} \int dx^4 Tr(-F_{\mu\nu}^2) + ...$, the relation

$$g_{YM}^2 \sim g_s \tag{91}$$

between the gauge theory coupling with the constant value of the dilaton is suggested. More precisely, when writing the SYM action using the complex coupling τ that combines g_{YM} with the instanton angle θ_I, such that the Montonen-Olive symmetry is manifest, and also combine the axion and the dilaton fields into the complex field 60, the identification becomes (restoring the correct proportionality constants):

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta_I}{2\pi} + \frac{C_0}{2\pi} + \frac{i}{g_s} \tag{92}$$

▲ The parameter N of the $SU(N)$ gauge group is identified with the flux of the 5-form Ramond-Ramond field strength on the 5-sphere.

$$\int_{S^5} *F_5 = N \tag{93}$$

▲ AdS_5 and S^5 have the same radius R, with $R^2 \sim \alpha' \sqrt{\lambda}$, where $\lambda = g_{YM}^2 N$ is the t' Hooft coupling of the gauge theory. The dimensionless parameters of the theories are related by:

$$\begin{aligned} g_s &\sim \frac{\lambda}{N} \\ \frac{R^2}{\alpha'} &\sim \sqrt{\lambda} \end{aligned} \tag{94}$$

The Maldacena conjecture in its strong form, suggests that the correspondence holds for all values of N and $g_s \sim g_{YM}^2$. That would involve the full Type II_B string theory on the $AdS_5 \times S^5$. However, string quantization on such a background is highly non-trivial and in order to deal with that, we rather consider some useful limits of the original statement.

5.2 The t' Hooft Limit

T' Hooft proposed that an $SU(N)$ Yang Mills gauge theory has a well defined perturbative expansion in the parameter $\frac{1}{N}$, if we take the number of colours N to be large, while keeping the coupling λ fixed:

$$N \to \infty \ , \quad g_{YM} \to 0 \ , \quad \lambda \equiv g_{YM}^2 N : \ fixed. \tag{95}$$

The graphical representation of this expansion constitutes of Feynman diagrams in the double line notation
($\to$: fundamental rep., $\leftarrow$: antifundamental rep., $\rightleftharpoons$: adjoint rep.). The computation of each one of them involves:

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{96}$$

and for $N \to \infty$, it is dominated by the Riemann surfaces of minimal genus g. By looking at equation 59, we see that the above expression is equivalent to the loop expansion in string theory, with string coupling $g_s \sim \lambda/N$.

Therefore, the large N limit of the $\mathcal{N} = 4$ conformal Super Yang Mills theory corresponds to the classical Type II_B string theory on $AdS_5 \times S^5$.

5.3 The Large λ Limit

One further limit, is to take the parameter λ to be large as well. This can still be achived even when $g_s \to 0$, so that the string loops are suprresed. Then, $\alpha'/R^2 \to 0$ and the higher derivative terms, as well as the massive string modes, are also suppresed (remember that higher derivative expansion is essentially an α' expansion), leaving us with an effective supergravity. From the field theory side, λ needs to be $\ll 1$ for the perturbation theory to be well defined. In other words, the weak coupling regime of the string theory corresponds to the strong coupling regime of the field theory. This strong/weak coupling duality exhibits the usefulness of the AdS/CFT correspondence. So, in the limit:

$$N \to \infty \ , \quad \lambda \to \infty \ , \quad g_{YM} \gg 1 \tag{97}$$

$\mathcal{N} = 4$ conformal Super Yang Mills theory corresponds to the classical Type II_B supergravity on $AdS_5 \times S^5$ and the correlation functions of the strongly coupled field theory can be computed using classical supergravity. When N and λ are finite, we rather turn into the full string theory. The field theory perturbative expasions are taken either in $1/N$, corresponding to string loop corrections in string theory, or in $1/\sqrt{\lambda}$, corresponding to α' corrections. For every order in $1/N$, there are infinite number of Feynman diagrams in field theory, leading to an extremely complicated funvtion $f_g(\lambda)$. In supergravity, however, every order is associated with only one graph and for $g = 0$ (planar field theory graphs) and λ large, the function $f_g(\lambda)$ can be calculated. Further corrections in $1/\sqrt{\lambda}$ correspond to string loop corrections, while going beyond the weak g_s coupling regime, requires world-sheet corrections.

5.4 Mapping The Symmetries

The most important check of the AdS/CFT correspondence is that both theories have the same global unbroken symmetries.

▲ The maximal bosonic subgroup of the conformal $\mathcal{N} = 4,\ d = 4$ Super Yang Mills is $SU(2,2) \times SU(4)_R \sim SO(2,4) \times SO(6)_R$, where $SU(2,2) \sim SO(2,4)$ is associated with the Poincare symmetry in 4 dimensions, combined with the conformal symmetry and $SU(4)_R \sim SO(6)_R$ is the R-symmetry group that rotates the supercharges into one another. In the string theory side, $SO(2,4)$ is by definition the isometry group of AdS_5 spacetime, while $SO(6)$ is associated with the rotations on the 5-sphere.

▲ This subgroup is enhanced to the full $SU(2,2|4)$ supergroup of the CFT, by addding to the 16 supersymmetries, the 16 conformal supersymmetries. On the other hand, D3-branes have 16 supercharges, being 1/2 BPS objects, which are enhanced to 32, as the near horizon AdS_5 space is a maximally sypersymmetric space. Thus, both theories have 32 supersymmetries.

▲ Furthermore, $\mathcal{N} = 4$ SYM has the discrete, global Montonen-Olive or S-duality symmetry, which is manifest when the coupling g_{YM} and the instanton angle θ are combined into the complex coupling

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta_I}{2\pi}.$$

Except from the invariance under $\theta \to 2\pi,\ \tau \to \tau + 1$ of the theory, there is also a $\tau \to -1/\tau$ invariance and those two together yield the S-duality symmetry group:

$SL(2,\mathbb{Z})$, with generator:

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \qquad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \tag{98}$$

In the AdS side, the $SL(2,\mathbb{Z})$ symmetry is manifest when using the complex field:

$$\tau \equiv \frac{C_0}{2\pi} + \frac{i}{g_s}.$$

Note that in the 't Hooft limit, S-duality becomes inconsistent, as, for $\theta_I = 0$, it maps $g_{YM} \to 1/g_{YM}$ and $\lambda \to N^2/\lambda$.

5.5 Mapping the CFT Operators to the Type II$_B$ Fields

The elementary fields of the conformal theory are not renormalized, so the spectrum is rather specified by a set of gauge invariant composite operators. Any irreducible conformal supermultiplet consists of one superconformal primary operator O, which is the operator of lowest dimension, and of its higher dimensional descendants O', that arise when applying the supercharges Q, ($[Q] = 1/2$), to O:

$$O' = [Q, O]_{\pm} \tag{99}$$

Therefore, a superconformal primary operator can not be given by the commutator of any other operator with Q. Keeping that in mind and looking at the supersymmetry transformations of the elementary fields, we conclude that the superconformal primary operators can only invole the scalar fields (without any derivatives or scalar commutators) arranged in a way that ensures gauge invariance. For that reason, we introduce the single trace operators of the form:

$$str\left(\Phi^{i_1}\Phi^{i_2}\cdots\Phi^{i_n}\right) \,, \quad str: \; symmetrized\ trace \tag{100}$$

They are totally symmetric in the indeces of the $SO(6)_R$ fundamental representation and they can be decomposed into irreducible operators, by isolating the the traces. The traceless part is then, denoted by $\{i_1...i_n\}$. In the simplest cases, we have:

$$\begin{aligned} &\sum_i Tr\ \Phi^i\Phi^i \\ &Tr\ \Phi^{\{i}\Phi^{j\}} \end{aligned} \tag{101}$$

The unitary representations of the $\mathcal{N}=4$ superconformal algebra, which are in one-to-one correspondence with the superconformal chiral primary operators, are labeled by the $SO(2,4)$ quantum numbers: (j_1, j_2): spin, Δ: conformal dimension and by the representations of $SU(4)_R$.

The conformal dimension of every operator in a unitary representation, satisfies a bound, set by the spin and the $SU(4)_R$ quantum numbers. For the scalar primary operators, there are four forms of the bound, which were worked out in [44]. What we are particularly interested in, is a classification of the multiplets into two kinds:

1 Chiral (or BPS) multiplets: These multiplets contain primary operators that are annihilated by some combination of the supercharges. Their length is shortened, as they include less conformal primary fields than the generic representations. An interesting feature of the chiral multiplets is that their conformal dimensions Δ are unrenormalised.

2 Non-Chiral (or non-BPS multiplets): Here, the primary operators do not commute with any of the supercharges. The dimensions Δ take continuous values, as opposed to the chiral ones, and they are not protected from quantum corrections.

The simplest case of a BPS operator is the 1/2 BPS operator, in the $[0, k, 0]$, $k \geq 2$ $SU(4)_R$ representation. It preserves 1/2 of the supersymmetries, its conformal dimension is k and has the general form:

$$O_\Delta(x) = \frac{1}{n_k} str \left(\Phi^{\{i_1}(x) \cdots \Phi^{i_k\}}(x) \right) \quad : \quad 1/2 \; BPS \tag{102}$$

Out of the above single trace operator, we can construct multiple trace 1/2 BPS operators and out of the latter, we can further constract 1/4 and 1/8 BPS operators.

In order to find the field contents of the $SU(2,2|4)$ irreducible representations on the AdS side, we perform a Kaluza-Klein compactification of the 10-dimensional space onto the compact manifold S^5. Then, the 10-dimensional supergravity fields are decomposed in an expansion on the 5-sphere. For the scalars, for instance, we write:

$$\phi(x, y) = \sum_{\Delta}^{\infty} \phi_\Delta(x) Y_\Delta(y), \tag{103}$$

where $\phi(x, y)$ are the 10-dimensional fields, with the coordinates x, y parametrizing the AdS_5 and the S^5 spaces respectively, Y_Δ is a basis for the spherical harmonics on the sphere and $\phi_\Delta(x)$ are the 5-dimensional fields living on AdS_5. The independent $\phi_\Delta(x)$ are combinations of Kaluza-Klein modes of different 10-dimensional bulk fields and they transform in representations of the $SO(6)$ group of rotations on the sphere. The full spectrum consists of a graviton multiplet and an infinite number of KK modes [45, 46, 47]. Each mode receives a mass contribution, which is determined by the rank Δ of the totally symmetric traceless representations of $SO(6)$. For the scalars for exaple, the relation between the mass of the fields and the conformal dimension is:

$$m^2R^2 = \Delta(\Delta - 4)$$

For the rest of the fields, see Table 2. Every KK mode falls into an $SU(2,2|4)$ representation and we obtain the same chiral multiplets as the ones of $\mathcal{N} = 4$ SYM. Therefore, there is an 1-1 correspondence between the fields from the Type II_B reduced supergravity and the chiral operators of $\mathcal{N} = 4$ SYM. Single trace operators are associated with single particles on ADS, while multiple trace operators, which can be formed by the single traced ones using OPE, are associated with bound states.

▲ 1/2 BPS supergravity excitations with a span of spin 2 correspond to chiral primary operators of the form: $O_2 = Tr\Phi^{\{i}\Phi^{j\}}$ plus descendents.

▲ 1/2 BPS supergravity KK excitations with a span of spin 2 correspond to chiral primary operators of the form: $O_\Delta = Tr\Phi^{\{i_1}...\Phi^{i_\Delta\}}$ plus descendents.

▲ non-BPS Type II_B massive string modes correspond to non-chiral operators like: $Tr\Phi^i\Phi^i$

▲ multiparticle states correspond to products of operators at different points: $O_{\Delta_1}(x_1)...O_{\Delta_n}(x_n)$

▲ bound states correspond to products of operators at the same point: $O_{\Delta_1}(x)...O_{\Delta_n}(x)$

6 Correlation Functions

6.1 Holographic Principle

Our aim is to relate two theories of different dimensions. More specifically, we are about to show a correspondence between 4-dimensional objects (operators) in a Yang-Mills theory and 5-dimensional objects (fields) in supergravity. This attempt is consistent with the so called "holographic principle" [22, 23], according to which, a quantum theory of gravity in a spacetime with dimensions d, can be described by an other theory that lives on the boundary of that space, thus having dimensions $d-1$. Moreover, the degrees of freedom of the boundary theory need to be less than one per Planck area, such that the entropy satisfies the Bekenstein bound [24]. This bound states that the maximum entropy of a space is proportional to the area of its boundary.

$$S \leq \frac{Area}{4G_N}$$

The important ingredient of holography that is used in our case of interest, is that the number of degrees of freedom within a region of spacetime, grows with the area of its boundary and not with its volume. Therefore, physics on AdS_5 can be captured by a local field theory that lives on its 4-dimensional Minkowskian boundary. But the fact that the field theory has infinte number of degrees of freedom, as it is conformal, and the Minkowskian boundary where it lives, is infinite as well, makes counting the degrees of freedom per Planck area preblematic. To deal with this, we introduce an energy cutoff in the field theory which corresponds to a cutoff in the radial distance in AdS_5. Then, sending this cutoff to zero, takes us to the horizon of AdS_5 and to the UV regime of the field theory.

6.2 The Bulk-Boundary Correspondance

Even though the supergravity low energy description of string theory, which is valid at large N and large 't Hooft coupling λ, is associated with the weakest form of the AdS/CFT correspondence, it is this limit that we work in, in order to calculate the planar contributions to n-point correlation functions of the Super Yang Mills gauge theory operators.

On the supergravity side, all 10-dimensional fields are decomposed onto Kaluza-Klein towers on S_5. We will be considering the 5-dimensional fields $\phi(z, x_\mu)$, which we shall call bulk fields, that live on AdS_5 and their dynamics is described by an effective action S_{AdS_5}.

The basic assumption of the correspondence is that every bulk field is associated with a single trace operator O [34, 36, 37, 38, 39], that belongs to the spectrum of the 4-dimensional Super Yang Mills theory. They both have the same $SO(2,4)$ quantum numbers and there is a relation between the mass m of the field and the scale dimension Δ of the operator. For scalar fields in particular, this relation is:

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$$

We can view the value of the bulk field on the boundary as a source for the corresponding operator, producing its correlation functions. More precisely, we add in the field theory Lagrangian an interaction term between the operator and the boundary field, identified with the generating functional for connected correlation functions of O. The extension of the source to the bulk is supported if the 5-dimensional field is a solution of the equations of motion that follow from the extremization of S_{AdS_5}. The mathematical expression of the above statetment is:

$$e^{W_{CFT}[\phi_0]} = \langle e^{-\int d^4x \;\, \phi_0(x) O(x)} \rangle_{CFT} \simeq e^{S_{AdS_5}[\phi]}|_{\phi(boundary)=\phi_0} \tag{104}$$

By differentiating $W_{CFT} \simeq S_{AdS_5}$ with respect to the sources ϕ_i^0 and then setting $\phi_i^0 = 0$, we get the planar contributions to the n-point connected correlation functions for the gauge invariant field operators.

$$\langle O_1 \cdots O_n \rangle = (-1)^{n-1} \frac{\delta}{\delta \phi_1^0} \cdots \frac{\delta}{\delta \phi_n^0} W_{CFT} \mid_{\phi_i^0 = 0} \tag{105}$$

Note that, since the sources are set to zero after the differentiation, interaction terms in the action with more than n fields will not contribute to the computation of the n-point function. Explicit calculations involve the expansion of both sides of expression 106 in a power series with respect to the sources. This semi-classical expansion can be represented in terms of tree level diagrams for the supergravity fields, the so called Witten diagrams.

A Witten diagram is essentially a disk, with the interior representing the AdS bulk and the boundary circle representing the boundary of AdS. A 4-diamensional source living on the circle, can be extended to a point in the bulk via a boundary-to-bulk propagator K, while vertex points in the interior of AdS, which represent interaction terms in S_{AdS_5}, can be connected via a bulk-to-bulk propagator G.

6.3 Bulk Fields

We refer to the fields that live on the 4-dimensional Minkowskian boundary of AdS_5 as boundary fields. On one hand, as already mentioned, they act as sources for the primary operators which specify the spectrum of the conformal field theory. On the other hand, they constitute they limit value of the AdS_5 bulk fields on its boundary and they can be extended to the full 5-dimensional configurations, $\phi(x_\mu) \to \tilde{\phi}(x_\mu, x_5)$ with $\mu = 0, 1, 2, 3$, when the appropriate boundary condiotions are imposed.

We will work with the Euclidean version of AdS_5, with the metric given in Poincare coordinates and the boundary corresponding to $z = 0$ (see equations 45, 50). The radious R is set equal to one for simplicity.

$$ds^2 = \frac{dx_\mu^2 + dz^2}{z^2} \tag{106}$$

Consider the effective action for a massive scalar field $\tilde{\phi}(x_\mu, z)$:

$$\begin{aligned} S &\sim \int dx^5 \sqrt{G} \left(G^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2 \right) \\ &= \int dz dx^\mu \frac{1}{z^5} \left(z^2 (\partial_z \phi)^2 + z^2 (\partial_\mu \phi)^2 + m^2 \phi^2 \right) \end{aligned} \tag{107}$$

The bulk fieds satisfy the equation of motion:

$$\partial_z \left(\frac{1}{z^3} \partial_z \phi \right) + \partial_\mu \left(\frac{1}{z^3} \partial^\mu \phi \right) = \frac{1}{z^5} m^2 \phi \tag{108}$$

Looking at the z-dependence only and writing the field as a power series, $\phi = \sum_{n=0}^{\infty} a_n z^n$, we get:

$$\begin{aligned} \sum_{n=0}^{\infty} \left(-3 a_n z^n + a_n n(n-1) z^n - m^2 a_n z^n \right) &= 0 \\ \Rightarrow n(n-4) &= m^2 \\ \Rightarrow n = 2 \pm \sqrt{4 + m^2} & \end{aligned} \tag{109}$$

Let us keep the largest value of this solution and call it Δ. Then,

$$\Delta(\Delta - 4) = m^2 \tag{110}$$

and the equation of motion has two linearly independent solutions:

$$\phi \sim \phi_0 z^{4-\Delta} + \phi_1 z^{\Delta} \tag{111}$$

Taking into account the x_μ dependence as well, we will just have some corrections depending on z and x_μ but the coefficients of ϕ_0, ϕ_1 will still have the same behaviour:

$$\phi(z, x_\mu) \sim \left[\phi_0(x_\mu) z^{4-\Delta} + O(z)\right] + \left[\phi_1(x_\mu) z^{\Delta} + O(z)\right] \tag{112}$$

What distinguishes the two solutions, is the property of renormalizability at the boundary $z = 0$, where the metric 106 blows up.

$$\int dx^5 \sqrt{G} |\phi|^2 \to \infty \tag{113}$$

The solution proportional to ϕ_0 is not properly squared normalizable. Thus, it does not correspond to bulk excitations; it is rather used us the coupling of external sources to the supergravity [25]. Notice that in order to get the five dimensional field configuration in the bulk we should set $\tilde{\phi}(z, x_\mu) \to f(z)\phi(x_\mu)$ instead of simply requiring $\tilde{\phi}(z, x_\mu) = \phi(x_\mu)$. The reason is that the equation of motion either diverges, for $\Delta > 4$, or vanishes, for $\Delta < 4$ at the boundary. And the only case where we get a constant value is when $\Delta = 4$. So, the necessary boundary condition is

$$\phi(z, x_\mu) \to z^{4-\Delta} \phi_0(x_\mu) \tag{114}$$

Given the value of the boundary field $\phi_0(x_\mu)$, $\phi_1(x_\mu, z)$ can be uniquely determined and we will obtail a complete solution for the bulk theory in AdS_5.
In order to deal with divergences that ocuur as $z \to 0$ while calculating several observables in AdS_5, it is appropriate to introduce a cutoff ϵ. Then the boundary conditions should be imposed at $z = \epsilon$ and in the end we may send this cutoff to zero. By doing this however, we do not have the full symmetry of AdS_5 anymore and thus, we can not find an exact solution in phase space. An exact expression can

be found instead, in terms of Bessel functions in momentum space p_μ, conjugate to x_μ. The Fourier transform of the 5-dimensional field is:

$$\phi(z, x_\mu) = \int dp^4 e^{ip_\mu x^\mu} \phi(z, p_\mu) \tag{115}$$

and the value at the cutoff is denoted by:

$$\phi_p(z = \epsilon) \equiv \phi_p^0 \epsilon^{4-\Delta} \tag{116}$$

The equation of motion 108 becomes

$$\left[(\partial_z)^2 - 3z\partial_z - (p^2 z^2 + m^2)\right] \phi(z, p_\mu) = 0 \tag{117}$$

which by setting $\phi_p = (pz)^2 f(pz)$, takes the form of a Bessel function

$$(pz)^2 \frac{d^2 f}{d(pz)^2} + (pz)\frac{df}{d(pz)} - \left(\Delta + (pz)^2\right) f = 0 \tag{118}$$

with two solutions: $z^2 I_{\Delta-2}(pz)$ and $z^2 K_{\Delta-2}(pz)$.

For $z \to 0$,

$$z^2 I_{\Delta-2}(pz) \to z^2 \left[\frac{(pz)}{2}\right]^{\Delta-2} \frac{1}{\Gamma(\Delta - 2 + 1)} \ \sim \ z^{\Delta},$$

while

$$z^2 K_{\Delta-2}(pz) \to z^2 \left[\frac{(pz)}{2}\right]^{2-\Delta} \frac{1}{\Gamma(\Delta - 2 + 1)} \ \sim \ z^{4-\Delta}.$$

The first solution increases exponentially as $z \to \infty$ and by requiring regularity in the deep interior, we need to exclude it. We only keep $z^2 K_{\Delta-2}(pz)$, the non renormalizable solution, which is exponentially vanishing as $z \to \infty$. Then, the normalized solution for the bulk field is:

$$\phi_p(z, p_\mu) = \frac{z^2 K_{\Delta-2}(pz)}{\epsilon^2 K_{\Delta-2}(p\epsilon)} \phi_p^0 \epsilon^{4-\Delta} \tag{119}$$

6.4 2-point Functions

Having found the solution to the supergravity equation of motion, subject to the appropriate Dirichlet boundary condition, we shall now evaluate the on-shell quadratic action, in order to obtain the generating functional $W_{CFT}[\phi_0] \simeq S_{AdS_5}|_{z=\epsilon}$ for the 2-point correlation function of the Yang Mills theory.
Integrating the action 110 by parts, yields:

$$S_{AdS_5} \sim \int_{AdS_5} \sqrt{G}\phi(-\Box + m^2)\phi + \int_{\partial AdS_5} \sqrt{G}\phi\partial^n\phi \tag{120}$$

Since the first term vanishes on the equation of motion, we are left with just the boundary contribution:

$$S_{AdS_5} \sim \frac{1}{z^3} \int dx^4 \phi \partial_z \phi \; |_{z=\epsilon} \tag{121}$$

Substituting ϕ from equation 115 and $\phi(p_\mu, z)$ from equation 119, we get:

$$
\begin{aligned}
S_{AdS_5} &\sim \frac{1}{z^3}\int dx^4 \int d^4p d^4q e^{ipx}\phi_p \partial_z e^{iqx}\phi_q \mid_{z=\epsilon} \\
&= \frac{1}{z^3}\int d^4p d^4q \; (2\pi)^4\delta^4(p+q)\phi_p\phi_q\partial_z(\log\phi_q) \mid_{z=\epsilon} \\
&= \frac{1}{z^3}\int d^4p d^4q \; (2\pi)^4\delta^4(p+q)\frac{z^2K_{\Delta-2}(pz)}{\epsilon^2K_{\Delta-2}(p\epsilon)}\phi_p^0\epsilon^{4-\Delta}\frac{z^2K_{\Delta-2}(qz)}{\epsilon^2K_{\Delta-2}(q\epsilon)}\phi_q^0\epsilon^{4-\Delta} \cdot \\
&\cdot \; \partial_z(\log\frac{z^2K_{\Delta-2}(qz)}{\epsilon^2K_{\Delta-2}(q\epsilon)}\phi_q^0\epsilon^{4-\Delta}) \mid_{z=\epsilon} \\
&= \frac{1}{z^3}\int d^4p d^4q \; (2\pi)^4\delta^4(p+q)\phi_p^0\epsilon^{4-\Delta}\phi_q^0\epsilon^{4-\Delta} \cdot \\
&\cdot \; \{\partial_z\log[z^2K_{\Delta-2}(qz)] + \partial_z\log(\phi_q^0\epsilon^{4-\Delta}) - \partial_z\log[\epsilon^2K_{\Delta-2}(q\epsilon)]\} \mid_{z=\epsilon} \\
&= \frac{1}{z^3}\int d^4p d^4q \; (2\pi)^4\delta^4(p+q)\phi_p^0\epsilon^{4-\Delta}\phi_q^0\epsilon^{4-\Delta}\cdot\partial_z\log[z^2K_{\Delta-2}(qz)] \mid_{z=\epsilon} \\
&\simeq W_{CFT}[\phi_0]
\end{aligned} \tag{122}
$$

The 2-point correlation function is then given by:

$$
\begin{aligned}
\langle O_\Delta(p)O_\Delta(q)\rangle &= -\frac{\delta}{\delta\phi_p^0}\frac{\delta}{\delta\phi_q^0}W_{CFT} \\
&= -\frac{\delta}{\delta\phi_p^0}\frac{\delta}{\delta\phi_q^0}\frac{1}{z^3}\int d^4p d^4q \; (2\pi)^4\delta^4(p+q)\phi_p^0\epsilon^{2(4-\Delta)}\phi_q^0\cdot\partial_z\log[z^2K_{\Delta-2}(qz)] \mid_{z=\epsilon} \\
&= -\frac{(2\pi)^4\delta^4(p+q)}{\epsilon^3}\epsilon^{2(4-\Delta)}\partial_\epsilon\log[\epsilon^2K_{\Delta-2}(q\epsilon)]
\end{aligned} \tag{123}
$$

For $\Delta - 2$ integer, which is the case in most applications of the AdS CFT correspondence, the asymptotic behavior of the Bessel function near $\epsilon = 0$, is of the form:

$$
\begin{aligned}
K_{\Delta-2}(q\epsilon) &= (q\epsilon)^{-(\Delta-2)}[a_0 + a_1(q\epsilon)^2 + a_2(q\epsilon)^4 + \ldots\,] + \\
&+ \; (q\epsilon)^{\Delta-2}\log(q\epsilon)[b_0 + b_1(q\epsilon)^2 + b_2(q\epsilon)^4 + \ldots\,]
\end{aligned}
$$

Then,

$$\begin{aligned}\langle O_\Delta(p)O_\Delta(q)\rangle &= \frac{(2\pi)^4\delta^4(p+q)}{\epsilon^4}\{-2+(\Delta-2)(1+c_2(q\epsilon)^2+c_4(q\epsilon)^4+\dots)-\\ &- \frac{2b_0(\Delta-2)}{a_0}(q\epsilon)^{2(\Delta-2)}\log(q\epsilon)(1+d_2(q\epsilon)^2+\dots)\}\end{aligned}$$

with $a_i,\ b_i,\ c_i,\ d_i$ being functions of $(\Delta-2)$ and $\frac{2b_0(\Delta-2)}{a_0}=\frac{(-1)^{(\Delta-2-1)}}{2^{2(\Delta-2-1)}\Gamma(\Delta-2)^2}$ [26].

The divergent terms that ocuur when we take the limit $\epsilon\to 0$, are local polynomials in q. In quantum field theory, these divergences are cured, by using appropriate local counter terms. Thus, as they are scheme dependent and physically unimportant, we can ignore them. We also drop the momentum conservation factor, while the two factors of $\epsilon^{4-\Delta}$ that originate from the boundary condition 114, are absorbed into the definition of the operators $O(p),\ O(q)$. We finally end up with the 2-point correlation function in momentum space:

$$\langle O_\Delta(p)O_\Delta(-p)\rangle = \frac{(-1)^{(\Delta-2)}}{2^{(2\Delta-6)}\Gamma(\Delta-2)^2}\ p^{2\Delta-4}\log p \tag{124}$$

The phase space correlator is obtained using differential regularization [27] or by analytic continuation in $(\Delta-2)$ [28, 29], so that the Fourier transform back to coordinate space can be defined.

$$\langle O_\Delta(x_1)O_\Delta(x_2)\rangle = \frac{(2\Delta-4)(\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}\frac{1}{|x_1-x_2|^{2\Delta}} \tag{125}$$

The above result is in agreement with the 2-point correlation function calculated using the Ward identities which involve the 3-point function $\langle O_\Delta(x_1)O_\Delta(x_2)J_\mu(x_3)\rangle$, where J_μ is a conserved current.

Restoring the AdS radius R, the relation 110 between the mass and the scale dimension of a scalar field becomes:

$$m^2R^2 = \Delta(\Delta - 4) \tag{126}$$

We see that positivity of energy, $m^2 > 0$, requires $\Delta > 4$, while the bound which is set by unitarity is just $\Delta \geq 1$. In fact, operators with scale dimensions $\Delta < 4$ exist and their energy is still positive, provided that the Breitenlohner-Freedman bound [30] $m^2R^2 \geq -4$ is satisfied.
We usually choose to define Δ as the largest solution of equation 109 because it is the normalizable one and it satisfies the unitarity bound. However, there are cases in which both solutions are normalizable and they give rise to correlators of two operators with different dimension. Then, the choice is made according to the transformation properties of the relevant fields under supersummetry or under a global bosonic symmetry [31, 32, 33].

The mass-dimension relation for fields of arbitary spin is shown in the table below [34, 35].

Table 2: Relation between mass and scale dimension

Fields	Dimension ($R = 1$)
scalars	$\Delta_\pm = \frac{1}{2}(d \pm \sqrt{d^2 + 4m^2})$
spinors	$\Delta = \frac{1}{2}(d + 2\vert m\vert)$
vectors	$\Delta_\pm = \frac{1}{2}(d \pm \sqrt{(d-2)^2 + 4m^2})$
p-forms	$\Delta_\pm = \frac{1}{2}(d \pm \sqrt{(d-2p)^2 + 4m^2})$
1st order $\frac{d}{2}-$forms, d:even	$\Delta = \frac{1}{2}(d + 2\vert m\vert)$
spin-3/2	$\Delta = \frac{1}{2}(d + 2\vert m\vert)$
massless spin-2	$\Delta = d$

6.5 AdS$_5$ Propagators

For the computation of the n-point functions with $n > 2$, it is more convinient to work in coordinate space, where the the conformal invariance is more obvious. Since higher order terms in the action are now taken into account,

$$S_{AdS_5} \sim \int dx^5 \sqrt{G} \left(\frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2} m_i^2 \phi_i^2 + \sum_{k=3}^{n} \lambda_{i_1 \ldots i_k} \phi_{i_1} \cdots \phi_{i_k} \right) \tag{127}$$

the equations of motion are no longer linear

$$\left(-\Box + m^2\right) \phi = \lambda \phi^n \tag{128}$$

and they can not be solved exactly. We therefore employ the iterative solution:

$$\phi^{zero}(z, x_\mu) = \int dx'_\mu K_\Delta(z, x_\mu - x'_\mu) \phi^0(x'_\mu) \tag{129}$$

$$\phi^{one}(z, x_\mu) = \lambda \int dx'_\mu G_\Delta(z - z', x_\mu - x'_\mu)[\phi^{zero}(z', x'_\mu)]^n \tag{130}$$

By inserting ϕ^{one} in the right hand side of 128, we find an expression for ϕ^{two} and we keep proceeding in this perturbative manner.

Equation 129 refers to the extension of a massive scalar field from the boundary point x_μ to a bulk point (z, x'_μ), which is now defined using the boundary-to-bulk propagator $K_\Delta(z, x_\mu - x'_\mu)$, satisfying the Klein-Gordon equation

$$(-\Box + m^2)K_\Delta(z, x_\mu - x'_\mu) = 0 \tag{131}$$

Its behavior at the boundary is specified by requiring:

$$K_\Delta(z, x_\mu - x'_\mu) \rightarrow z^{4-\Delta}\delta(x_\mu - x'_\mu)\ , \quad z \rightarrow 0 \tag{132}$$

The above limit suggests that, as we approach the AdS boundary, K_Δ looks like a delta function at $z = \infty$. Notice that the solution z^Δ of the Klein-Gordon equation, see equation 114, is zero on all of the boundary except at that specific point, where it is infinity. As mentioned in section 3.4 , in order to obtain the full 4-dimensional spherical boundary of the compactified Euclidean AdS_5, the point $z = \infty$ had to be added. The unique solution of 131 that also satisfies 132, is:

$$K_\Delta(z, x_\mu - x'_\mu) = \frac{\Gamma(\Delta)}{\pi^2\Gamma(\Delta - 2)}\left(\frac{z}{z^2 + (x_\mu - x'_\mu)^2}\right)^\Delta \tag{133}$$

When considering interaction terms, we need the bulk-to-bulk propagator that appears in equation 130. It solves the equation:

$$(-\Box + m^2)G(z - z', x_\mu - x'_\mu) = \frac{1}{\sqrt{G}}\delta(z - z')\delta(x_\mu - x'_\mu) \tag{134}$$

and it is given by the hypergeometric function [40, 12]:

$$G_\Delta(u) = \frac{\Gamma(\Delta)\Gamma(\Delta-3/2)}{(4\pi)^{5/2}\Gamma(2\Delta-3)}(2u^{-1})^\Delta F\left(\Delta,\Delta-\frac{7}{2};2\Delta-3;-2u^{-1}\right), \quad (135)$$

$$u = \frac{[(z,x_\mu)-(z',x'_\mu)]^2}{2zz'}$$

6.6 3-point Functions

For the calculation of the 3-point function, the action 127 for three scalar fields, reduces to:

$$S_{AdS_5} \sim \int dx^5\sqrt{G}\left(\frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}m_i^2\phi_i^2 + \lambda\phi_1\phi_2\phi_3\right) \quad (136)$$

There is only one interaction term which corresponds to one cubic vertex in the bulk. Therefore, there is only one graph, including three boundary-to-bulk propagators. The correlation function for the associated operators reads:

$$\begin{aligned}\langle O_{\Delta_1}(x_1)O_{\Delta_2}(x_2)O_{\Delta_3}(x_3)\rangle &= -\lambda\int dx^5\sqrt{G}K_{\Delta_1}(z,x-x_1)K_{\Delta_2}(z,x-x_2)K_{\Delta_3}(z,x-x_3) \\ &= \frac{\lambda a_1}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}} \quad (137)\end{aligned}$$

Evaluation of the x-integral [28] determines the constant a_1.

$$\begin{aligned}a_1 = &- \frac{\Gamma[(\Delta_1+\Delta_2-\Delta_3)/2]\Gamma[(\Delta_1+\Delta_3-\Delta_2)/2]\Gamma[(\Delta_2+\Delta_3-\Delta_1)/2]}{2\pi^4\Gamma(\Delta_1-2)\Gamma(\Delta_2-2)\Gamma(\Delta_3-2)} \cdot \\ &\cdot \Gamma[(\Delta_1+\Delta_2+\Delta_3)/2] \quad (138)\end{aligned}$$

The cubic couplings of the supergravity fields in $AdS_5 \times S_5$ are obtained by expanding in fluctuations around the background metric and the F_5 field strength [43].

7 Conclusions

Dualities between string theories and quantum field theories through different approaches, seem to provide valuable ways of extracting information about the two sides. The importance of AdS/CFT correspondece lies in the insight given on the strongly coupled Yang Mills theories by the weak/strong duality. In the interesting limit of large N and strong coupling, the field theory's observables can receive a successful perturbative treatment using a classical gravitational theory.

Another important point is that the correspondence is realised holographically, the theories involved differ in the number of spacetime dimensions by 1, with one theory lying on the boundary of the other. Because of the infinite amount of degrees of freedom, associated with conformal invariance, in order to check the holographic principle is satisfied, we need to introduce a cutoff δ. In the field theory side, this cutoff is some energy scale, whereas in the gravity side it is related to a radial position. Then, going from the interior of AdS to the boundary by sending this cutoff to zero, corresponds to going to the UV regime of the field theory. Considering for example the $\mathcal{N} = 4$ SYM on the S^3 unit sphere, the number of d.o.f goes like $\sim N^2\delta^{-3}$. On the other hand, the area of a surface in AdS, for $\delta \to 0$, in Planck units is:

$$\frac{Area}{4G_N} = \frac{V_{S^5}R^3\delta^{-3}}{4G_N} \sim N^2\delta^{-3}$$

Therefore, we conclude that AdS/CFT correspondece satisfies the holographic bound. The statement that any field theory- AdS duality is holographic should be understood as checking whether, for different values of R (i.e. for different values of N), the number of d.o.f. goes like the area of the space or like its volume. Such a holographic description is usefull for considering aspects of black holes. We would expect it to provide information about the suitable treatment of black holes singularities. It moreover concludes that the evolution of black holes is unitary, as the theory on the boundary is unitary.

The main purpose of this thesis, was the introduction to the Maldacena's conjecure, that states the correspondence between $\mathcal{N} = 4$ Superconformal YM and Type II_B string theory on $AdS_5 \times S^5$ background, with the identifications: $\frac{g_{YM}^2}{4\pi} = g_s^2$ and $R^2 = \alpha' \sqrt{4\pi g_s N}$. This is the original, strong form of the conjecture. However, by also considering limits where it becomes more useful, it can be summarized into the following three forms:

- strong form $\rightarrow$ all values of N and $g_s \sim g_{YM}^2 \rightarrow$ full string theory.
- 't Hooft limit $\rightarrow \lambda = g_{YM}^2 N$: fixed, $N\infty$ $\rightarrow$ classical string theory on $AdS_5 \times S^5$.
- weak limit$\rightarrow \lambda \rightarrow \infty$, $N \rightarrow \infty$ classical supergravity on $AdS_5 \times S^5$.

In order to make the above arguments manifest, we started by studying the geometry of AdS_5. The key features of that space are:

1 It has maximal supersymmetry, i.e. 32 supercharges

2 It has, by constraction, an $SO(2, 4)$ isometry, which is the conformal group in 4 dimensions.

3 The boundary of the conformally compactified AdS_5 is 4-dimensional Minkowski spacetime.

Then, 10-dimensional Type II_B superstring and supergravity theories were dicussed and in that content, p-branes were introduced as 1/2 BPS solutions to the supergravity equations of motion, preserving half of the supersymmetries of the background and they were extended to the notion of Dp-branes in string theory, were open strings can end. The key points in that section were:

1 Thinking of the Dp-branes as sources embedded in the Type II_B background, they cause a deformation of the geometry in the near horizon region, which, in the p=3 case, corresponds to $AdS_5 \times S^5$. In the low energy limit the flat spacetime in the region far away from the branes decouples and we are left with just the near horizon curved geometry.

2 On the world volume of a set of N coincident 3-branes, lives an $SU(N)$ vector supermultiplet with 16 supersymmetries, which are enhanced to 32 when we add the conformal supersymmetries. This is associated with the fact that the near horizon region is AdS.

Having motivated the AdS/CFT correspondence, we then moved on to the matching of the parameters, the symmetries and the observables, finishing with a demonstration of how to calculate the Yang Mills correlation functions of composite operators, using the boundary fields of AdS space as sources, formulating a generating functional that is actually, the sypergravity 5-dimensional effective action, evaluated on the equations of motion. It turns out that the 1-, 2- and 3-point functions satisfy the constraints that follow from conformal invariance, while 4-point functions are lees constrained.

Since the AdS/CFT correspondence was first introduced there has been a large number of ckecks on it, with remarkably successful results. However, there are some points to be stressed about its possible extentions.
The systems that have been taken under consideration, have a UV fixed point but they are not characterized by asymptotical freedom. The key point of the correspondence that has drawn all this attention, is the duality between weak and strong coupling. The problem is, that if a theory is weaky coupled at high energies, then the AdS' curvature is large and the full string theoretical approach is required, something that is not yet achievable. Recent considerations that attempt

to overcome this problem, suggest the introduction of a UV cutoff in the geometry, such that QCD dynamics could be approached by phenomenological models.

Another significant direction in which future studies need to be focused, is the extension of the corespondence to theories with less symmetries. In non supersymmetric Yang Mills theories, the field content is much smaller and we do not have a way to treat the extra fields. Furthermore, there are attempts to introduce fundamental fields into the correspondence, in order to describe QCD, as $\mathcal{N} = 4$ SYM only contains the vector supermultiplet, in which all fields are transformed in the adjoint representation.

Let us finish by pointing out the need for the AdS/CFT correspondence to be extended to theories that are very symmetric in the UV regime but in the IR regime they deform into theories with less symmetries. Conformal symmetry is given up, in order to get descrete mass spectra, leading to more interesting theories, from a phenomenological point of view.

References

[1] G. t Hooft, *Nucl. Phys.* **B72** (1974).

[2] J. M. Maldacena.*Adv. Theor. Math. Phys.* **2** (1998).

[3] C. Burgess and C. Lutken, *Phys. Lett.* **1153B** (1985) 137.

[4] T. Inami and H. Ooguri, *Prog. Theor. Phys.* **73** (1985) 1051.

[5] E. Witten, *Nucl. Phys.* **B460** (1996) 335-350.

[6] G. T. Horowitz and A. Strominger, *Nucl. Phys.* **B360** (1991) 197-209.

[7] F. Gliozzi, J. Scherk and D. I. Olive, *Nucl. Phys.* **B122** (1977) 253.

[8] W. Nahm, *Nucl. Phys.* **B 135** (1978) 149.

[9] E. Cremmer, B. Julia and J. Scherk, *Phys. Lett.* **B76** (1978) 409.

[10] G. Dall'Agata, K. Lechner and M. Tonin, arXiv:hep-th/9812170v1 (1998).

[11] G. Dall'Agata, K. Lechner and M. Tonin, arXiv:hep-th/9806140v2 (1998).

[12] E. D'Hoker and D. Z. Freedman, arXiv:hep-th/0201253v2 (2002)

[13] J. Polchinski, String Theory vol.2 Cambridge University Press (1998) 531.

[14] P. S. Howe and P. C. West, *Phys. Lett.* **B126** (1983) 301.

[15] D. Garfinkle, G. T. Horowitz abd A. Strominger , *Phys. Rev.* **D43** (1991) 3140.

[16] J. Dai, R. G. Leigh and J. Polchinski, *Mod. Phys. Lett.* **A4** (1989) 2073.

[17] J. Polchinski, *Phy. Rev. Lett.* **75**, (1995) 4724.

[18] J. Polchinski, S. Chaudhuri and C. V. Johnson, arXiv:hep-th/9602052.

[19] C. G. Callan, C. Lovelace, C. R. Nappi and Yost, *Nucl. Phys.* **B308**, (1988) 221.

[20] C. G. Callan, C. R. Nappi and S. A. Yost, *Nucl. Phys.* **280**, (1987) 599.

[21] E. Witten, *Nucl. Phys.* **B500**, (1997) 3.

[22] L. Susskind, *J. Math. Phys.* **36**, (1995) 6377-6396.

[23] G. t' Hooft, gr-qc/9310026.

[24] J. D. Bekenstein, *Phys. Rev.***D49**, (1994) 1912-1921.

[25] V. Balasubramanian and P. Kraus, *Phys. Rev. Lett.***83**, (1999) 3605.

[26] I. S. Gradshteyn and I. M. Ryzhik, Academic Press, New York, (1965).

[27] D. Z. Freedman,K. Johnson and J. I. Latorre, *Nucl. Phys.* **B371**, (1992) 353.

[28] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, *Nucl. Phys.* **B546**, (1999) 96.

[29] H. Liu and A. A. Tseytlin, *Nucl. Phys.* **B533**, (1998) 88.

[30] P. Breitenlohner and D. Z. Freedman, *Phys. Lett.* **B115**, (1982) 197.

[31] S.W. Hawking, *Phys. Lett.* **B126**, (1983) 175.

[32] L. Mezincescu and P. K. Townsend, *Phys. Lett.* **B148**, (1984) 55.

[33] I. R. Klebanov and E. Witten, *Nucl.Phys.* **B556**, (1999) 89.

[34] E. Witten, *Adv. Theor. Math. Phys.* **2**, (1998) 253.

[35] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, hep-th/9904017.

[36] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Phys. Lett.* **B428**, (1998).

[37] G. E. Arutyonov and S. A. Frolov, *Nucl. Phys.* **B544**, (1999).

[38] W. Muck and K. S. Viswanata, *Phys. Rev.* **D60**, (1999).

[39] H. Boschi-Filho and N. R. Braga, *Phys. Lett.* **B505**, (2001).

[40] O. Aharony, S. S. Gubser, J. M. Maldestana, H. Ooguri and Y. Oz, *Phys. Rept.* **323**, (2000).

[41] J. H. Schwarz, *Nucl. Phys.* **B226**, (1983).

[42] P. S. Howe and P. C. West, *Phys. Lett.* **B238** (1984) 181.

[43] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Adv. Theor. Math. Phys.* **2** (1999).

[44] V. K. Dobrev and V. B. Petkova, *Lett. Math. Phys.* **9**, (1985) 287.

[45] O. Aharony, Y. Oz and Z. Yin, . *Phys. Lett.* **B430**, (1998).

[46] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen *Phys. Rev.* **D32**, 1985.

[47] M. Gunaydin and N. Marcus, *Class. Quant. Grav.* **2** L11, (1985).